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MAT 175 Spring 2010: Exam 2
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04/23/10

Name: Solution

Instruction: Please read the questions carefully. You must write complete solutions to receive complete credit.

1. (10 points) Find all eigenvalues and a basis of each eigenspace of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ y-z \\ 2y+4z \end{bmatrix}$.

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$. $A - \lambda I_3 = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{bmatrix}$

$$\det(A-\lambda I_3) = (2-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} = (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big)$$
$$= (2-\lambda)(4-5\lambda+\lambda^2+2) = 0$$
$$= (2-\lambda)(\lambda-2)(\lambda-3)$$
$$= -(\lambda-2)^2(\lambda-3).$$

The eigenvalues are $\lambda = 2, \lambda = 3$.

$E_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\to y = 0, z = 0$

x is a free variable

$= \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

A basis of E_2 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

$E_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$: $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$= \text{Span}\left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$. A basis of E_3 is $\left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$.

2. (10 points) Let $\vec{w}$, $\vec{z}$ be vectors in $\mathbb{R}^n$. Let

$$U = \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \cdot \vec{w} = 0 \text{ and } \vec{u} \cdot \vec{z} = 0\}.$$

Show that U is a subspace of $\mathbb{R}^n$.

(1) $\vec{0}$ is in U.

Since $\vec{0} \cdot \vec{w} = 0$ and $\vec{0} \cdot \vec{z} = 0$, it follows that $\vec{0} \in U$.

(2) U is closed under vector addition

Let $\vec{u}, \vec{v}$ be ~~vectors in U. $\vec{v} \cdot \vec{w} = 0$, $\vec{v} \cdot \vec{z} = 0$~~

~~Since $\vec{u} \cdot \vec{w} = 0$ and $\vec{u} \cdot \vec{z} = 0$ we then have that~~

~~$(\vec{u}+\vec{v}) \cdot \vec{w}$~~ Since $\vec{u} \cdot \vec{w} = 0$, $\vec{u} \cdot \vec{z} = 0$, $\vec{v} \cdot \vec{w} = 0$, and $\vec{v} \cdot \vec{z} = 0$, these imply that

$(\vec{u}+\vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0 + 0 = 0$, and

$(\vec{u}+\vec{v}) \cdot \vec{z} = \vec{u} \cdot \vec{z} + \vec{v} \cdot \vec{z} = 0 + 0 = 0$.

Hence, $\vec{u} + \vec{v} \in U$.

(3) U is closed under scalar multiplication.

Let $\vec{u} \in U$ and let α be a scalar.

Since $\vec{u} \cdot \vec{w} = 0$ and $\vec{u} \cdot \vec{z} = 0$, we then have that

$(\alpha\vec{u}) \cdot \vec{w} = \alpha(\vec{u} \cdot \vec{w}) = \alpha(0) = 0$, and

$(\alpha\vec{u}) \cdot \vec{z} = \alpha(\vec{u} \cdot \vec{z}) = \alpha(0) = 0$. So $\alpha\vec{u} \in U$.

By (1)-(3), we can conclude that U is a subspace of $\mathbb{R}^n$.

3. (10 points) Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ be a linearly independent subset of $\mathbb{R}^3$. Please use the Gram-Schmidt process to transform a set $\mathcal{B}$ into an orthonormal basis of $\mathbb{R}^3$.

Hint: $\vec{v}_i = \vec{u}_i - \frac{\vec{u}_i \cdot \vec{v}_1}{||\vec{v}_1||^2}\vec{v}_1 - \frac{\vec{u}_i \cdot \vec{v}_2}{||\vec{v}_2||^2}\vec{v}_2 - \ldots - \frac{\vec{u}_i \cdot \vec{v}_{i-1}}{||\vec{v}_{i-1}||^2}\vec{v}_{i-1}$.

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\|\vec{v}_1\|^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3.$$

$$\vec{u}_2 \cdot \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\vec{u}_3 \cdot \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$$

$$\vec{u}_3 \cdot \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 1/3$$

$$\|\vec{v}_2\|^2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1/3}{2/3}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\frac{1/3}{2/3} = 1/2$$

$$= \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} -2/3(1/2) \\ 1/3(1/2) \\ 1/3(1/2) \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/6 \\ -3/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$\|\vec{v}_3\|^2 = \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \end{bmatrix} = 0 + \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$$

An orthonormal basis is $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2/3}}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \frac{1}{\sqrt{1/2}}\begin{bmatrix} 0 \\ -1/2 \\ -1/2 \end{bmatrix} \right\}$.

4. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y+z \\ x-4y \\ 3x \end{bmatrix}.$$

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^3$. Find $[T]_{\mathcal{B}}$ (the matrix representation of T with respect to $\mathcal{B}$). $[T]_{\mathcal{B}} = B^{-1}AB$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}.$$

$$T(\vec{b}_1) = T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}, \quad T(\vec{b}_2) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$T(\vec{b}_3) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 1 & 1 & 0 & -3 & -3 & 1 \\ 1 & 0 & 0 & 3 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 3 \\ 0 & 1 & 0 & -6 & -6 & -2 \\ 0 & 0 & 1 & 6 & 5 & -1 \end{array}\right].$$

5. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ 2 & 1 & 4 \\ 0 & 2 & -4 \end{bmatrix}$. Find bases for

(a) the column space of A,(5 points)

(b) the null space of A. (5 points)

(c) Extend a basis of the null space of A to a basis of the whole space $\mathbb{R}^3$. (5 points)

Since $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ 2 & 1 & 4 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we can conclude that

a) $Col(A) = Span\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and a basis of $col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$,

b) $Null(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\vdots$ $\begin{matrix} x + 3z = 0 \\ y - 2z = 0 \end{matrix}$.

$= \left\{ \begin{bmatrix} -3z \\ 2z \\ z \end{bmatrix} \middle| z \in \mathbb{R} \right\} = Span\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$,

and a basis of $Null(A) = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

c) Since $\begin{bmatrix} -3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we can conclude that

$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$ and $\mathcal{B}$ contains $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

6. (10 points) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & c \end{bmatrix}$. Assume that the characteristic polynomial of A is $-(t-c)(t-3)(t-4)$. Determine all values of the scalar c for which A is not diagonalizable.

If $c \neq 3$ and $c \neq 4$ then A is diagonalizable.

When $c = 3$, we have

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ 6 & 6-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 & 0 \\ 6 & 6-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

and the characteristic polynomial of A is $-(t-3)^2(t-4)$.

This implies that $\dim E_4 = 1$.

$$E_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| (A - 3I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} -2 & -1 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

$= \text{Span}\left\{ \begin{bmatrix} -1/2\, y \\ y \\ z \end{bmatrix} \middle| y, z \in \mathbb{R} \right\}$ since $\begin{bmatrix} -2 & -1 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \text{Span}\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. So $\dim E_3 = 2$

So, A is diagonalizable when $c = 3$.

When $c = 4$, we have that the characteristic polynomial when $c = 4$ of A is $-(t-3)(t-4)^2$ and $\dim E_3 = 1$.

$E_4 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} -3 & -1 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Since $\begin{bmatrix} -3 & -1 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

we can conclude that $\dim E_4 = 2$. Hence, A is diagonalizable